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ON THE ALMOST-SURE ASYMPTOTIC STABILITY OF SECOND ORDER LINEAR STOCHASTIC SYSTEM

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A method for obtaining a sufficient almost-sure stability condition for second order linear systems with an ergodic stiffness coefficient is presented. In this method, a special Lyapunov function for achieving functional optimization is constructed and the probabilistic property of the derivative process of the stiffness is taken into account. A sufficient condition for almost-sure asymptotic stability is derived and numerical results are presented for the cases of Gaussian noise and periodic noise coefficient. The results obtained here are an improvement over previously available results for linear systems with stochastic stiffness.

1. INTRODUCTION

The almost-sure (abbreviated as a.s.) asymptotic stability of the trivial solution of the second order system in the form

$$\ddot{x} + 2[\zeta + f(t)]\dot{x} + [1 + g(t)]x = 0,$$

where $f(t)$ and $g(t)$ are ergodic random processes, has attracted intensive study in the last three decades. When $f(t)$ and $g(t)$ are ergodic wide-band Gaussian processes which may be approximated by white noise processes, the almost-sure asymptotic stability of the system has been considered by Mitchell and Kozin [1], who employed a method of Khas'minskii's [2] to obtain numerically the exact stability boundary. When $f(t)$ and $g(t)$ are arbitrary ergodic random processes, Kozin and Wu [3] took into account the distribution properties of the coefficient processes and obtained sufficient a.s. asymptotic stability boundaries numerically, which enabled them to obtain much sharper results than those obtained by Infante [4]. Using the optimization method, Ariaratnam and Xie [5, 6] have considered the asymptotic stability boundaries of the system in the cases when $f(t)$ and $g(t)$ are uncorrelated or correlated. It is known, however, that knowledge of more statistic information on the processes $f(t)$ and $g(t)$ should enable one to get sharper and sufficient a.s. asymptotic stability boundaries. In fact, Ariaratnam and Xie [7] took the derivative process of $f(t)$, $\dot{f}(t)$ into account and enlarged the asymptotic stability boundaries in the case when $g(t)$ equals zero. In this paper, the a.s. asymptotic stability of the system is considered in the case when $f(t)$ equals zero. We applied a special Lyapunov function with a view to achieving functional optimization that takes the derivative process of $g(t)$, $\dot{g}(t)$ into account.

It is shown that the asymptotic stability boundaries can be significantly enlarged for Gaussian and periodic excitation. Numerical results and comparison with previous results are presented.

2. FORMULATION

Consider the following second order system:

$$\ddot{x} + 2\zeta\dot{x} + [1 + g(t)]x = 0, \tag{1}$$

where $\zeta \geqslant 0$ is a damping coefficient, and $g(t)$ is a stationary ergodic differentiable process with zero mean value.

In previous studies [1, 3–6], the probabilistic properties of the derivative process $\dot{g}(t)$ were not taken into account. In order to bring in the influence of the derivative process $\dot{g}(t)$, a transformation of the following form is considered:

$$x = y\mathrm{e}^{-\zeta t} \tag{2}$$

which, when substituted into equation (1), yields

$$\ddot{y} + [c + g(t)]y = 0, \tag{3}$$

where $c = 1 - \zeta^2$.

Equation (3) can be written in the state equation forms as

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= -[c + g(t)]y_1. \end{aligned} \tag{4}$$

It may be noted that in equation (3) the damping term has been removed, which is a significant change, since for this case the norm of y, $\|y\|$ (Lyapunov function), may be given in the very simple form

$$\|y\|^2 = V(y) = y^{\mathrm{T}}Ay, \tag{5}$$

where A is a positive-definite matrix of the form

$$A = \begin{bmatrix} \alpha^2(t) & 0 \\ 0 & 1 \end{bmatrix}, \tag{6}$$

where $\alpha(t) \geqslant \delta > 0$ is a stochastic process to be determined and δ is a constant. The time derivative of V along the trajectories of equation (4) yields

$$\dot{V}(y) = y^{\mathrm{T}} B y, \tag{7}$$

where

$$B = \begin{bmatrix} 2\dot{\alpha}(t)\alpha(t) & a^2(t) - c - g(t) \\ a^2(t) - c - g(t) & 0 \end{bmatrix}. \tag{8}$$

Therefore, since A, B are real symmetric matrices and A is positive definite, one has

$$\frac{\dot{V}}{V} = \frac{y^{\mathrm{T}} B y}{y^{\mathrm{T}} A y} \leqslant \lambda(BA^{-1}), \tag{9}$$

where λ is the maximum eigenvalue of BA^{-1}, i.e., λ is the maximum root of the determinant equation

$$|B - \lambda A| = 0. \tag{10}$$

Equation (9) yields

$$V[y(t)] \leqslant V_0 \exp\left\{\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\},$$

where $V_0 = V[y(0)]$. Since $\|y\|^2 = V$, there is

$$\|y\|^2 \leqslant \|y_0\|^2 \exp\left\{\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\} = \|y_0\|^2 \exp\left\{t\frac{1}{t}\int_0^t \lambda(\tau)\,\mathrm{d}\tau\right\}. \tag{11}$$

For stationary ergodic process $g(t)$ it is supposed, in a tradition followed in previous studies, that $\lambda(t)$ is also a stationary ergodic process. When $t \to +\infty$, the right-hand side of equation (11) goes to $\|y_0\|^2 \exp\{tE[\lambda(t)]\}$ with probability 1 (w.p.1). Therefore from equation (2) there is w.p.1 as $t \to +\infty$,

$$|x|^2 = |y_1|^2 \exp\{-2\zeta t\} \leqslant \frac{1}{\delta^2}\|y\|^2 \exp\{-2\zeta t\} \leqslant \frac{1}{\delta^2}\|y_0\|^2 \exp\{tE[\lambda(t)] - 2\zeta t\},$$

where $|\cdot|$ denotes the absolute value.

Therefore as $t \to +\infty$, w.p.1,

$$|x| \leqslant \frac{1}{\delta}\|y_0\| \exp\left\{\frac{t}{2}[-2\zeta + E[\lambda(t)]]\right\}. \tag{12}$$

Letting

$$\lambda^* = \tfrac{1}{2}[-2\zeta + E[\lambda(t)]],$$

a sufficient condition for a.s. asymptotic stability of the trivial solution of equation (1) is given by

$$\lambda^* \leqslant -\varepsilon, \quad \varepsilon > 0$$

or

$$-2\zeta + E[\lambda(t)] \leqslant -\varepsilon. \tag{13}$$

Substituting matrices A, B from equations (6) and (8) into equation (10), one obtains

$$|B - \lambda A| = \begin{bmatrix} -\lambda\alpha^2(t) + 2\dot{\alpha}(t)\alpha(t) & a^2(t) - c - g(t) \\ a^2(t) - c - g(t) & -\lambda \end{bmatrix} = 0,$$

i.e.,

$$\lambda^2\alpha^2(t) - 2\alpha(t)\dot{\alpha}(t)\lambda - [\alpha^2(t) - c - g(t)]^2 = 0.$$

Therefore, its maximum eigenvalue is

$$\lambda = \frac{1}{\alpha(t)}\{\dot{\alpha}(t) + \sqrt{\dot{\alpha}^2(t) + [\alpha^2(t) - c - g(t)]^2}\}. \tag{14}$$

Substituting equation (14) into equation (13), it follows that the trial solution of equation (1) is asymptotic stable w.p.1 if

$$-2\zeta + E\left\{\frac{1}{\alpha(t)}\{\dot{\alpha}(t) + \sqrt{\dot{\alpha}^2(t) + [\alpha^2(t) - c - g(t)]^2}\}\right\} \leqslant -\varepsilon. \tag{15}$$

Since

$$\sqrt{\dot{\alpha}^2(t) + [\alpha^2(t) - c - g(t)]^2} \leqslant |\dot{\alpha}(t)| + |\alpha^2(t) - c - g(t)|$$

condition (15) can be relaxed to the following condition:

$$-2\zeta + E\left\{\frac{1}{\alpha(t)}\{\dot{\alpha}(t) + |\dot{\alpha}(t) + |\alpha^2(t) - c - g(t)|\}\right\} \leqslant -\varepsilon. \tag{16}$$

If $\alpha(t)$ is chosen as constant as in the studies cited [1, 3–7], $\dot{\alpha}(t) = 0$, then condition (16) becomes

$$E|\alpha^2 - c - g(t)| \leqslant 2\zeta\alpha - \varepsilon, \tag{17}$$

where α is a constant to be determined to get the sharpest stability boundary.

However, the right-hand side of inequality (17) should be $2\zeta\alpha - \alpha\varepsilon$; since ε is an arbitrary positive constant, it can be written in the form in inequality (17). In the following cases, ε may have different values in different equations. If the stability condition is desired in terms of $E\{g^2(t)\}$, applying Schwarz inequality, there is

$$E|\alpha^2 - c - g(t)| \leqslant \sqrt{E(\alpha^2 - c - g(t))^2}.$$

Then condition (17) may be relaxed as

$$E(\alpha^2 - c - g(t))^2 \leqslant 4\zeta^2\alpha^2 - \varepsilon.$$

Remembering that $Eg(t) = 0$, the above inequality yields

$$Eg^2(t) \leqslant -(\alpha^2 - \zeta^2 - 1)^2 + 4\zeta^2 - \varepsilon.$$

Since ε is an arbitrary positive constant, the above condition yields the following stability boundary:

$$Eg^2(t) = -(\alpha^2 - \zeta^2 - 1)^2 + 4\zeta^2.$$

Obviously, when $\alpha = \sqrt{\zeta^2 + 1}$, one obtains the maximum stability boundary

$$Eg^2(t) = 4\zeta^2$$

which is the same as that of Infante's [3]. In the same way, condition (17) yields the following stability boundary:

$$E|\alpha^2 - c - g(t)| = 2\zeta\alpha.$$

In previous studies α is a constant to be determined to get a sharper stability boundary with a view to achieving function optimization. However, with a view to achieving functional optimization, we can choose $\alpha(t)$ as a stochastic process to get a sharper stability boundary. It is best to choose $\alpha(t)$ in such a way that the left-hand side of inequality (16) reaches the minimum value. But such $\alpha(t)$ is not easy to obtain. Hence $\alpha(t)$ is chosen such that the following equation will reach the minimum value:

$$E\left\{\frac{1}{\alpha(t)}\{|\alpha^2(t) - c - g(t)|\}\right\} = \min.$$

Obviously, $\alpha(t) = \sqrt{|c + g(t)|}$. Since $\alpha(t)$ should be greater than some positive number, $\alpha(t)$ is selected to be

$$\alpha(t) = \begin{cases} \delta & \text{when } |c + g(t)| \leqslant \delta^2, \\ \sqrt{|c + g(t)|} & \text{when } |c + g(t)| > \delta^2, \end{cases} \tag{18}$$

where $\delta > 0$ is a constant to be determined. Equation (18) yields

$$\dot{\alpha}(t) = \begin{cases} -\dfrac{\dot{g}(t)}{2\sqrt{|c + g(t)|}} & \text{when } c + g(t) < -\delta^2, \\ 0 & \text{when } -\delta^2 \leqslant c + g(t) \leqslant \delta^2, \\ \dfrac{\dot{g}(t)}{2\sqrt{|c + g(t)|}} & \text{when } c + g(t) > \delta^2. \end{cases} \tag{19}$$

For process $g(t) = g(t, \omega)$, $\omega \in \Omega$, and Ω being the sample space, it is supposed that the set $G = \{(t, \omega) : |c + g(t, \omega)| = \delta\}$ is a zero-measure set; then $\alpha(t)$ is differentiated beside set G and equations can be derived without considering the case when $(t, \omega) \in G$. In fact, $\alpha(t)$ may be undifferentiated at the point "t" such that $|c + g(t)| = \delta$. Substituting equations (18) and (19) into inequality (16), the following a.s. condition is obtained:

$$-2\zeta + E\left\{\frac{|\dot{g}(t)| - \dot{g}(t)}{2|c + g(t)|}, c + g(t) < -\delta^2\right\} + E\left\{\frac{|\dot{g}(t)| + \dot{g}(t)}{2|c + g(t)|}, c + g(t) > \delta\right\}$$
$$+2E\{\sqrt{|c + g(t)|}, c + g(t) < -\delta^2\} + E\left\{\frac{\delta^2 - c - g(t)}{\delta}, \; -\delta^2 \leqslant c + g(t) \leqslant \delta^2\right\} \leqslant -\varepsilon, \tag{20}$$

where $\delta > 0$ is a constant to be determined such that the left-hand side of inequality (20) reaches the minimum value.

If the probabilistic properties of $g(t)$ and $\dot{g}(t)$ are known, from condition (17) or condition (20), the stability boundary of the trivial solution of equation (1) can be obtained. By combining conditions (17) and condition (20), a *shaper* stability boundary can be obtained.

In the following section, some specific examples are given.

3. EXAMPLES

In this section, the results of conditions (17) and (20) are applied to the case where the noise coefficient is Gaussian and to the case where the coefficient is a cosine function with random phase. For these cases numerical computations are required, and the approach is described below.

3.1. EXAMPLE 1

Consider the case when $g(t)$ is a zero mean Gaussian process. The density function of $g(t)$ is

$$p(g) = \frac{1}{\sqrt{2\pi}\sigma_g} \exp\left\{-\frac{g^2}{2\sigma_g^2}\right\}. \tag{21}$$

From equation (17) the stability boundary is obtained as

$$E|\alpha^2 - c - g(t)| = 2\zeta\alpha,$$

i.e.,

$$\int_{-\infty}^{+\infty} \left|\alpha^2 - c - g\right| \frac{1}{\sqrt{2\pi}\sigma_g} \exp\left\{-\frac{g^2}{2\sigma_g^2}\right\} \mathrm{d}g = 2\zeta\alpha. \tag{22}$$

In order to obtain the best-possible condition, the parameter α is considered to be not only a function of stiffness ζ but also a function of σ_g. By using the optimization numerical computation method, a suitable α is chosen such that σ_g reaches the maximum value for a given ζ. The results of the numerical computation are shown in Fig. 1 by curve 3.

Since $g(t)$ is a Gaussian process, the derivative process $\dot{g}(t)$ is also a zero mean Gaussian process. The density function of $\dot{g}(t)$ is

$$p(\dot{g}) = \frac{1}{\sqrt{2\pi}\sigma_{\dot{g}}} \exp\left\{-\frac{g^2}{2\sigma_{\dot{g}}^2}\right\}. \tag{23}$$

Since $E\{g(t)\dot{g}(t)\} = 0$ and $g(t)$ and $\dot{g}(t)$ are independent, there is

$$\begin{aligned} E&\left\{\frac{|\dot{g}(t)| - \dot{g}(t)}{2|c + g(t)|}, c + g(t) < -\delta^2\right\} \\ &= E\{|\dot{g}(t)| - \dot{g}(t)\} E\left\{\frac{1}{2|c + g(t)|}, c + g(t) < -\delta^2\right\} \\ &= E\{|\dot{g}(t)|\} E\left\{\frac{1}{2|c + g(t)|}, c + g(t) < -\delta^2\right\} \\ &= \frac{\sigma_{\dot{g}}}{\sqrt{2\pi}} \int_{-\infty}^{-\delta^2 - c} \frac{1}{|c + g|} \frac{1}{\sqrt{2\pi}\sigma_g} \exp\left\{-\frac{g^2}{2\sigma_g^2}\right\} \mathrm{d}g \end{aligned} \tag{24}$$

Similarly,

$$\begin{aligned} E&\left\{\frac{|\dot{g}(t)| + \dot{g}(t)}{2|c + g(t)|}, c + g(t) > \delta^2\right\} \\ &= \frac{\sigma_{\dot{g}}}{\sqrt{2\pi}} \int_{\delta^2 - c}^{+\infty} \frac{1}{c + g} \frac{1}{\sqrt{2\pi}\sigma_g} \exp\left\{-\frac{g^2}{2\sigma_g^2}\right\} \mathrm{d}g \end{aligned} \tag{25}$$

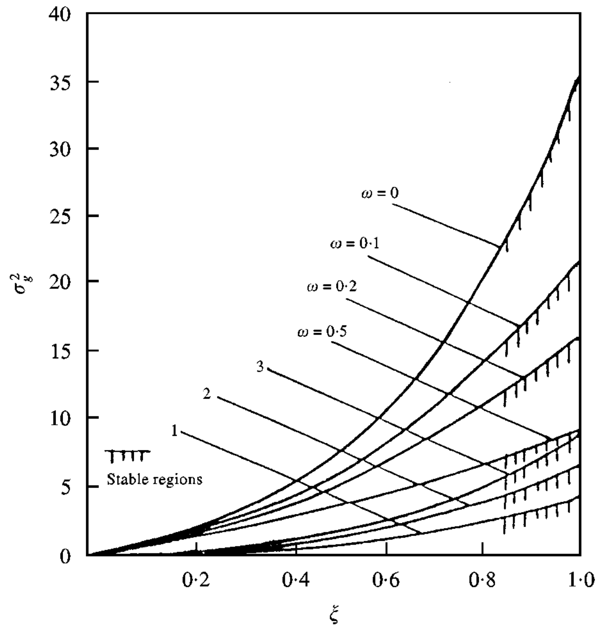

Figure 1. Regions of a.s. asymptotic stability for $\ddot{x} + 2\zeta\dot{x} + [1 + g(t)]x = 0$, curve: (1) via Infante [4], (2) via Kozin and Wu [3], (3) via equation (22).

Substituting equations (24) and (25) into equation (20), a sufficient asymptotic stability boundary is obtained

$$\frac{\sigma_{\dot{g}}}{\sqrt{2\pi}}\left[\int_{-\infty}^{-\delta^2-c} + \int_{\delta^2-c}^{+\infty}\right]\frac{1}{|c+g|}\frac{1}{\sqrt{2\pi}\sigma_g}\exp\left\{-\frac{g^2}{2\sigma_g^2}\right\}\mathrm{d}g$$

$$+ 2\int_{-\infty}^{-\delta^2-c}\sqrt{|c+g|}\frac{1}{\sqrt{2\pi}\sigma_g}\exp\left\{-\frac{g^2}{2\sigma_g^2}\right\}\mathrm{d}g$$

$$+ \int_{-\delta^2-c}^{\delta^2-c}\frac{\delta^2-c-g}{\delta}\frac{1}{\sqrt{2\pi}\sigma_g}\exp\left\{-\frac{g^2}{2\sigma_g^2}\right\}\mathrm{d}g = 2\zeta, \tag{26}$$

where $\delta > 0$ is a constant to be determined. Let $\omega = \sigma_{\dot{g}}/\sigma_g$ for given ω and ζ, and $\delta > 0$ is chosen such that σ_g reaches the maximum value. For different $\omega(\omega = 0, 0{\cdot}1, 0{\cdot}2, 0{\cdot}5)$, the results of the numerical computation are shown in Fig. 1 by the curves $\omega = 0, 0{\cdot}1, 0{\cdot}2, 0{\cdot}5$ respectively.

The results of Kozin and Wu [3] and Infante [4], which are sufficient asymptotic stability conditions without taking $\sigma_{\dot{g}}$ into account, are represented in Fig. 1 by curves 2 and 1, respectively, for comparing with the present results. It is obvious that for small values of ω, the results from equation (26) are the best; the results from equation (22), which are independent of ω, are better than those of Kozin and Wu [3] and Infante [4]. For large values of ω, the results of equation (26) may not be as good as those of Kozin and Wu [3] and Infante [4], but the results of equation (22) are also better than those of Kozin and Wu [3] and Infante [4]. Hence, the improved results by combining equations (22) and (26) are better than those of Kozin and Wu [3] and Infante [4].

3.2. EXAMPLE 2

Consider the case when $g(t)$ is a periodic coefficient

$$g(t) = A\cos(\omega t + \theta), \tag{27}$$

where $A \geqslant 0$, $\omega \geqslant 0$ are fixed amplitude and frequency, and θ is a uniformly distribution random phase on the interval $[0, 2\pi]$.

The density function for this process is

$$p(g) = \frac{1}{\pi\sqrt{A^2 - g^2}}, \quad |g| < A \tag{28}$$

and is independent of ω or θ.

However, in equation (28) g is a variable and does not present the stochastic process $g(t)$, which is defined by equation (27). And for arbitrary t, $g(t)$ has the same density function $p(g)$ which is independent of the random variable $g(t)$.

From condition (17), the following stability boundary can be obtained:

$$E|\alpha^2 - c - g(t)| = \int_{-A}^{A} \frac{|\alpha^2 - c - g(t)|}{\pi\sqrt{A^2 - g^2}}\,\mathrm{d}g = 2\zeta\alpha.$$

Let $u = g/A$; the above equation yields

$$\int_{-1}^{1} \frac{|\alpha^2 - c - Au|}{\pi\sqrt{1 - u^2}}\,\mathrm{d}u = 2\zeta\alpha, \tag{29}$$

where $\alpha > 0$ is a constant to be determined such that the amplitude A reaches the maximum value.

The way to choose α can be discussed in the following three cases.

In the case when $|\alpha^2 - c| \leqslant A$

$$\int_{-1}^{1} \frac{|\alpha^2 - c - Au|}{\pi\sqrt{1 - u^2}}\,\mathrm{d}u = -\int_{(\alpha^2 - c)/A}^{1} \frac{\alpha^2 - c - Au}{\pi\sqrt{1 - u^2}}\,\mathrm{d}u + \int_{-1}^{(\alpha^2 - c)/A} \frac{\alpha^2 - c - Au}{\pi\sqrt{1 - u^2}}\,\mathrm{d}u$$

$$= 2\left[\frac{\alpha^2 - c}{\pi}\arcsin\frac{\alpha^2 - c}{A} + \frac{A}{\pi}\sqrt{1 - \left(\frac{\alpha^2 - c}{A}\right)^2}\right] = 2\zeta\alpha. \tag{30}$$

Let

$$t = \frac{\alpha^2 - c}{A} \qquad h(t) = t \arcsin t + \sqrt{1 - t^2}.$$

From the calculation, one can choose $\alpha > 0$ such that the amplitude A reaches the maximum value

$$A_1 = \max A = \max_{t \in D} \left\{ \frac{\pi^2 \zeta^2 t}{4h^2(t)} + \sqrt{\frac{\pi^4 \zeta^4 t}{4h^4(t)} + \frac{\pi^2 \zeta^2 (1 - \zeta^2)}{h^2(t)}} \right\}, \tag{31}$$

where the domain D is

$$D = \{t: -1 \leqslant t \leqslant 1, \pi^2 \zeta^2 t^2 + 4(1 - \zeta^2) h^2(t) \geqslant 0\}. \tag{32}$$

Obviously, domain D is not a null set; in fact $t = 1$ is a point in the domain D, and hence the maximum value A_1 defined by equations (30) and (31) exists.

In the case when $\alpha^2 - c > A$, equation (29) yields

$$\int_{-1}^{1} \frac{|\alpha^2 - c - Au|}{\pi \sqrt{1 - u^2}} \mathrm{d}u = \int_{-1}^{1} \frac{\alpha^2 - c - Au}{\pi \sqrt{1 - u^2}} \mathrm{d}u = \alpha^2 - c = 2\zeta\alpha,$$

i.e., $(\alpha - \zeta)^2 = 1$. Let $\alpha = \zeta + 1$, the maximum value of A is

$$\max A = 2\zeta + 2\zeta^2.$$

In the case when $\alpha^2 - c < -A$, $c = 1 - \zeta^2 > 0$, equation (29) yields

$$\int_{-1}^{1} \frac{|\alpha^2 - c - Au|}{\pi \sqrt{1 - u^2}} \mathrm{d}u = -\int_{-1}^{1} \frac{\alpha^2 - c - Au}{\pi \sqrt{1 - u^2}} \mathrm{d}u = -\alpha^2 + c = 2\zeta\alpha,$$

i.e., $(\alpha + \zeta)^2 = 1$. Let $\alpha = 1 - \zeta$, the maximum value of A is

$$\max A = 2\zeta - 2\zeta^2.$$

To sum up the above three cases, the maximum value of A can be obtained from equation (17)

$$\max A = \max\{2\zeta + 2\zeta^2, A_1\}, \tag{33}$$

where A_1 is defined by equation (31).

Equation (27) yields

$$\dot{g}(t) = -A \sin(\omega t + \theta), \qquad |\dot{g}(t)| = \omega \sqrt{A^2 - g^2(t)}. \tag{34}$$

Using the symmetric property of $\sin(\omega t+\theta)$ and $\cos(\omega t+\theta)$, there is

$$E\left\{\frac{\dot g(t)}{2|c+g(t)|}, c+g(t)>\delta^2\right\} = E\left\{\frac{\dot g(t)}{2|c+g(t)|}, c+g(t)<-\delta^2\right\} = 0. \tag{35}$$

Substituting equations (34) and (35) into equation (20), the following stability boundary is obtained:

$$\begin{aligned}
&\frac{\omega}{2}E\left\{\frac{\sqrt{A^2-g^2(t)}}{2|c+g(t)|}, c+g(t)<-\delta^2\right\} + \frac{\omega}{2}E\left\{\frac{\sqrt{A^2-g^2(t)}}{c+g(t)}, c+g(t)>\delta^2\right\}\\
&\quad + 2E\{\sqrt{|c+g(t)|}, c+g(t)<-\delta^2\} + E\left\{\frac{\delta^2-c-g(t)}{\delta},\ -\delta^2\leqslant c+g(t)\leqslant\delta^2\right\}\\
&\quad = 2\zeta,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\frac{\omega}{2}\int_{\delta^2-c}^{A}\frac{\sqrt{A^2-g^2}}{c+g}\frac{\mathrm{d}g}{\pi\sqrt{A^2-g^2}} - \frac{\omega}{2}\int_{-A}^{-\delta^2-c}\frac{\sqrt{A^2-g^2}}{c+g}\frac{\mathrm{d}g}{\pi\sqrt{A^2-g^2}}\\
&\quad + 2\int_{-A}^{-\delta^2-c}\sqrt{|c+g|}\frac{\mathrm{d}g}{\pi\sqrt{A^2-g^2}} + \int_{-\delta^2-c}^{\delta^2-c}\frac{\delta^2-c-g}{\delta}\frac{\mathrm{d}g}{\pi\sqrt{A^2-g^2}}\\
&\quad = 2\zeta.
\end{aligned}$$

After calculation, the above equation results in

$$\begin{aligned}
&\frac{\omega}{2}\ln\left|\frac{(c+A)(c-A)}{[c+(\delta^2-c)][c+(-\delta^2-c)]}\right| + 2\int_{-A}^{-\delta^2-c}\sqrt{|c+g|}\frac{\mathrm{d}g}{\pi\sqrt{A^2-g^2}}\\
&\quad + \frac{(\delta^2-c)}{\delta}\left(\arcsin\frac{(\delta^2-c)}{A} - \arcsin\frac{(-\delta^2-c)}{A}\right)\\
&\quad + \frac{1}{\delta}[\sqrt{A^2-(\delta^2-c)^2} - \sqrt{A^2-(-\delta^2-c)^2}] = 2\zeta\pi,
\end{aligned}\tag{36}$$

where $\delta>0$ is constant to be determined such that the amplitude A reaches the maximum value. In the case when $-\delta^2-c<-A$, or $\delta^2-c<-A$, one should only substitute all terms of $(-\delta^2-c)$ or (δ^2-c) in equation (36) into $-A$. In the case when $-\delta^2-c>A$, or $\delta^2-c>A$, one should only substitute all terms of $(-\delta^2-c)$ or (δ^2-c) in equation (36) into A.

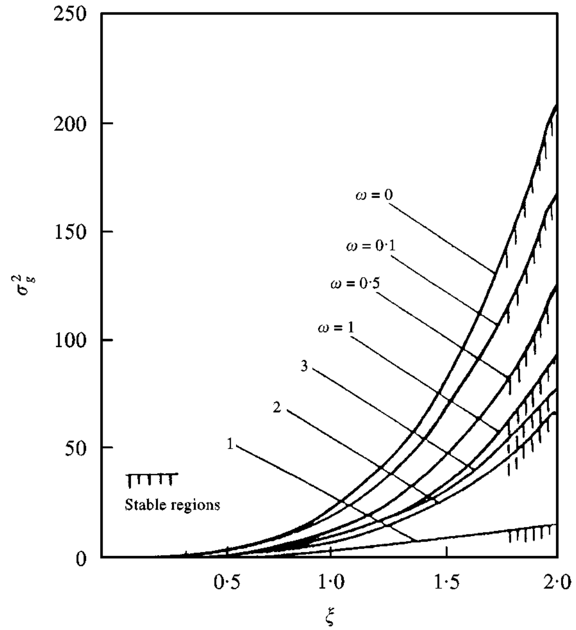

Figure 2. Regions of a.s. asymptotic stability for $\ddot{x} + 2\zeta\dot{x} + [1 + g(t)]x = 0$, curve: (1) via Infante [4], (2) via Kozin and Wu [3], (3) via equation (33).

For different ω and ζ, $\delta > 0$ is selected such that the amplitude A reaches the maximum value. Obviously,

$$\sigma_g^2 = E\{g^2(t)\} = E\{A^2\cos^2(\omega t + \theta)\} = \frac{A^2}{2},$$

$$\sigma_{\dot{g}}^2 = E\{\dot{g}^2(t)\} = \frac{A^2}{2}\,\omega^2,$$

$$\omega = \frac{\sigma_{\dot{g}}}{\sigma_g}.$$

For different $\omega(=0, 0{\cdot}1, 0{\cdot}2, 0{\cdot}5, 1)$, the results of the numerical computation are shown in Fig. 2 by the curves $\omega = 0$, 0·1, 0·2, 0·5, 1 respectively.

The results of Kozin and Wu [3] and Infante [4], which are sufficient asymptotic stability conditions without taking $\sigma_{\dot{g}}$ into account, are represented in Fig. 2 by curves 2 and 1, respectively, for comparing with the present results. It is obvious that for small values of ω, the results from equation (36) are the best; the results from equation (33), which are independent of ω, are better than those of Kozin and Wu [3] and Infante [4]. For large values of ω, the results of Equation (36) may not be as good as those of Kozin and Wu [3] and Infante [4], but the results of equation (33) are also better than those of Kozin and Wu [3] and Infante [4]. Hence, the improved results by combining equations (33) and (36) are better than those of Kozin and Wu [3] and Infante [4].

4. CONCLUSION

By using a specific Lyapunov function on the view of functional optimization, the method of obtaining a sufficient a.s. condition for second order linear systems with an ergodic stiffness coefficient has been presented, which also takes into account the probabilistic property of the derivative process of the stiffness coefficient. A sufficient condition for stability has been derived and numerical results have been presented for the cases of a Gaussian noise coefficient and periodic noise coefficient. The results have been found to be an improvement over those in the literature for systems with a stochastic stiffness coefficient.

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